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AN ALGORITHM FOR MAGIC TESSERACTS

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Magic squares and cubes have fascinated people throughout centuries. A generalization of magic squares and magic cubes are magic tesseracts (magic four-dimensional cube). In Andrew's book [1] its are called *magic octahedroids*. By a *magic tesseract* of order n we mean a four-dimensional matrix

$$\mathbf{Q}_n = |\mathbf{q}(i_1, i_2, i_3, i_4);\ 1 \leq i_1, i_2, i_3, i_4 \leq n|,$$

containing natural numbers $1, \dots, n^4$ such that the sum of the numbers along every row and every diagonal is the same, i.e. $\frac{n(n^4+1)}{2}$.

By a *row* of $\mathbf{Q}_n$ we mean a 4-tuple of elements $\mathbf{q}(i_1, i_2, i_3, i_4)$ which have identical coordinates at 3 places. A *diagonal* of $\mathbf{Q}_n$ is a 4-tuple of elements $\{\mathbf{q}(x, i_2, i_3, i_4) : x = 1, \dots, n,\ i_j = x \ \text{ or } i_j = n+1-x \text{ for all } 2 \leq j \leq 4\}$. A magic tesseract $\mathbf{Q}_n$ contains $4n^3$ rows and 8 great diagonals. The symbol $\lfloor x \rfloor$ denotes the integral part of x, the symbol $x \pmod n$ denotes the number $x - n\lfloor \frac{x}{n} \rfloor$ and the symbol x^* denotes the minimum of the set $\{x, n+1-x\}$.

This paper contains formulas for construction a magic tesseract of order n for every $n \neq 2$. A similar algorithm for magic cubes is in [2].

We consider three cases:

$n \equiv 1 \pmod 2$, $\quad n \equiv 0 \pmod 4$ $\quad$ or $\quad n \equiv 2 \pmod 4$.

Figure 1 shows the nine layers of $\mathbf{Q}_3$. The element $\mathbf{q}(1,1,1,1) = 46$ is in four rows containing the triplets

$$\{46, 8, 69\}, \qquad \{46, 62, 15\}, \qquad \{46, 17, 60\}, \qquad \{46, 59, 18\}.$$

On the eight diagonals there are the triplets

$$\begin{array}{ll} \{\mathbf{q}(1,1,1,1) = 46, 41, 36\}, & \{\mathbf{q}(1,1,1,3) = 69, 41, 13\}, \\ \{\mathbf{q}(1,1,3,1) = 15, 41, 67\}, & \{\mathbf{q}(1,1,3,3) = 35, 41, 47\}, \\ \{\mathbf{q}(1,3,1,1) = 60, 41, 22\}, & \{\mathbf{q}(1,3,1,3) = 26, 41, 56\}, \\ \{\mathbf{q}(1,3,3,1) = 53, 41, 29\}, & \{\mathbf{q}(1,3,3,3) = 64, 41, 18\}. \end{array}$$

(Notes: 1. This picture is a magic square of order 9 with some special properties. 2. In [4] it is point out a relationship between the mathematical brain-twister SUDOKU and four-dimensional Latin hypercubes and magic tesseracts.

46	8	69	17	78	28	60	37	26
62	42	19	51	1	71	10	80	33
15	73	35	55	44	24	53	6	64
59	39	25	48	7	68	16	77	30
12	79	32	61	41	21	50	3	70
52	5	66	14	75	34	57	43	23
18	76	29	58	38	27	47	9	67
49	2	72	11	81	31	63	40	20
56	45	22	54	4	65	13	74	36

Figure 1 - Magic tesseract $\mathbf{Q}_3$

1. If n is an odd integer then a magic tesseract $\mathbf{Q}_n$ can be constructed using the following formula

$$\begin{aligned}\mathbf{q}(i_1,i_2,i_3,i_4) &= [(i_1-i_2+i_3-i_4+\frac{n-1}{2}) \pmod n]n^3\\ &+ [(i_1-i_2+i_3+i_4-\frac{n+3}{2}) \pmod n]n^2\\ &+ [(i_1-i_2-i_3-i_4+\frac{3n+1}{2}) \pmod n]n\\ &+ [(i_1+i_2+i_3+i_4-\frac{3n-1}{2}) \pmod n]+1.\end{aligned}$$

2. If $n \equiv 0 \pmod 4$ then

$$\mathbf{q}_n(i_1,i_2,i_3,i_4) = u[v \pmod 2] + (n^4+1-u)[(v+1) \pmod 2]$$

where

$$\begin{aligned}u &= (i_1-1)n^3+(i_2-1)n^2+(i_3-1)n+i_4,\\ v &= i_1+\lfloor\frac{2(i_1-1)}{n}\rfloor+i_2+\lfloor\frac{2(i_2-1)}{n}\rfloor+i_3+\lfloor\frac{2(i_3-1)}{n}\rfloor+i_4+\lfloor\frac{2(i_4-1)}{n}\rfloor.\end{aligned}$$

3. If $n \equiv 2 \pmod 4$ (in this case $t = \frac{n}{2}$ is odd) then

$$\mathbf{q}_n(i_1, i_2, i_3, i_4) = \mathbf{s}(u, v)t^4 + \mathbf{q}_t(i_1^*, i_2^*, i_3^*, i_4^*),$$

where $\mathbf{S} = |\mathbf{s}(u, v) : 1 \leq u \leq t, 1 \leq v \leq 16|$ is a matrix defined by the following tables.

	$\mathbf{s}(u,1)$	$\mathbf{s}(u,2)$	$\mathbf{s}(u,3)$	$\mathbf{s}(u,4)$	$\mathbf{s}(u,5)$	$\mathbf{s}(u,6)$	$\mathbf{s}(u,7)$	$\mathbf{s}(u,8)$
$\mathbf{s}(1,v)$	15	7	14	6	13	5	12	4
$\mathbf{s}(2,v)$	7	15	6	14	5	13	4	12
$\mathbf{s}(3,v)$	0	1	3	2	5	4	6	7
$\mathbf{s}(2a+2,v)$	0	1	2	3	4	5	6	7
$\mathbf{s}(2a+3,v)$	15	14	13	12	11	10	9	8

$\mathbf{s}(u,9)$	$\mathbf{s}(u,10)$	$\mathbf{s}(u,11)$	$\mathbf{s}(u,12)$	$\mathbf{s}(u,13)$	$\mathbf{s}(u,14)$	$\mathbf{s}(u,15)$	$\mathbf{s}(u,16)$
11	3	10	2	9	1	8	0
3	11	2	10	1	9	0	8
9	8	10	11	12	13	15	14
8	9	10	11	12	13	14	15
7	6	5	4	3	2	1	0

Table $\mathbf{S} = |\mathbf{s}(u, v)|$, $\quad a = 1, 2, \ldots, \frac{n-6}{4}$

- $u = (i_1^* - i_2^* + i_3^* - i_4^*) \pmod{\frac{n}{2}} + 1$,
- $v = 8\lfloor\frac{2i_1-1}{n}\rfloor + 4\lfloor\frac{2i_2-1}{n}\rfloor + 2\lfloor\frac{2i_3-1}{n}\rfloor + \lfloor\frac{2i_4-1}{n}\rfloor + 1$.

Note: The following three proprieties of $\mathbf{S}$ are very important for our construction.

1. Every row of $\mathbf{S}$ is the set $\{0, \ldots, 15\}$.
2. The sum of elements in v-th column is the same for every v such that the number of ones in the binary representation of the number $v - 1$ is even (or odd.)
3. $\mathbf{s}(1, v) + \mathbf{s}(1, 17 - v) = 15$ for $v = 1, \ldots, 8$.

References

[1] W.S.Andrews, *Magic Squares and Cubes*, Dover, New York 1960

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